Foliations and 2+1 Causal Dynamical Triangulation Models

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The original models of causal dynamical triangulations construct space-time by arranging a set of simplices in layers separated by a fixed time-like distance. The importance of the foliation structure in the $2+1$ dimensional model is studied by considering variations in which this property is relaxed. It turns out that the fixed-lapse condition can be equivalently replaced by a set of global constraints that have geometrical interpretation. On the other hand, the introduction of new types of simplices that puncture the foliating sheets in general leads to different low-energy behavior compared to the original model.

I. INTRODUCTION

Causal dynamical triangulation models are a non-perturbative approach to the study of quantum gravity. In recent years, these models have been used to construct a Lorentzian path integral for quantum gravity as a sum over geometries constructed by gluing many primary building blocks together [1]. A $1+1$ dimensional model has been fully solved analytically [2] and higher dimensional versions have been studied numerically [3, 4, 5]. Simulations in $3+1$ dimensions suggest that causal dynamical triangulations generate large-scale space-times with desirable properties [5].

Causal dynamical triangulations are based on the Regge action for simplicial gravity, where macroscopic space-times are constructed from elementary building blocks glued together. These elementary building blocks are $n-$simplices, with n ranging from zero to the dimensionality of the space-time. The Lorentzian nature of the space-time is implemented by making some of the edges in the simplices time-like. In the original models, the elementary building blocks come only in a few varieties [4]. The constructions give rise to foliated space-times in which every layer of simplices is separated from another layer by a space-like hypersurface formed by the faces of the simplices. Hence, there is a clear distinction between space-like and time-like directions.

The foliation structure has been argued to play an important role in giving the Lorentzian models desirable low-energy properties such as an appropriate Hausdorff dimension [4]. Given that the same properties do not arise in Euclidean models, it is interesting to ask how rigid the structure of the studied Lorentzian models really is. What are the consequences of loosening the assumptions in the original Lorentzian models, and is it possible to build more general models with the same low-energy properties? This paper address two aspects of this question.

One important aspect of the original Lorentzian model is the requirement for all time-like edges to have equal length-squared, which can be thought of as a 'lapse' constraint. In $1+1$ dimensions, this requirement on the individual simplices can be removed so that their shapes and sizes can be chosen quite freely as long as a global constraint on the ensemble of simplices making up the entire space-time is enforced instead [6]. One of the results of this paper is that a similar generalization is possible in higher dimensions, where the analogous global constraints can be interpreted as specifying average volume and/or curvature contributions for the simplices in the triangulation.

A second aspect of the original model is connected with the types of simplices that are used as the primary building blocks of space-time. The existence of global hyper-surfaces constructed from the space-like faces of simplices can be compromised by introducing new types of simplices into the triangulation. Introducing new types of simplices that puncture the hyper-surfaces foliating the space-time in general causes triangulation models to move into a different equivalence class than that of the original models. Therefore, the extended models may have different low energy properties than the original ones. The new models, however, can reproduce the familiar behavior if additional but ad-hoc constraints are introduced to restrict the number of foliation-puncturing simplices that appear in the triangulations.

This paper focuses on $2+1$ dimensional models for concreteness but, where possible, the discussion is extended to higher dimensions. Section II is a short summary of the original Lorentzian model in $2+1$ dimensions and includes a discussion of its associated partition function. Section III discusses how the lapse can be allowed to vary. The following section IV deals with the new types of simplices that puncture the foliation hypersurfaces. A summary of the findings appears in section V.

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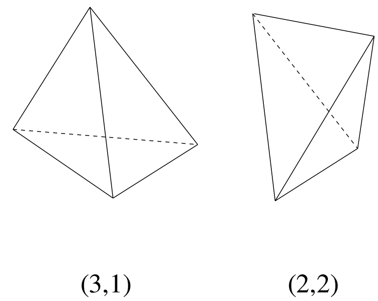

FIG. 1: Simplices in 2+1 dimensions. Edges pointing mostly vertically (mostly sideways) are time-like (space-like) and have negative (positive) edge-lengths. Simplices are named after the number of vertices on space-like surfaces.

II. ORIGINAL MODEL

In Lorentzian dynamical triangulation models in $2+1$ dimensions, a set of tetrahedra are glued together along their faces to construct an extended space-time. The original model uses the $(3,1)$ and $(2,2)$ types of tetrahedra shown in figure 1. This section reviews the original $2+1$ model; for more details, see [4].

In the original model, all space-like edges have length-squared set to unity, and all time-like edges have length-squared equal to $-\alpha$, where α is a positive constant. Space-times formed by gluing these simplices together always form a layered structure and contain space-like surfaces formed by the space-like faces of the $(3,1)$ tetrahedra. These surfaces can be thought of as the discrete analogs of foliation hyper-surfaces. The Regge action corresponding to such a configuration is

$$S = k_0 \sum_a \left(-iV(a)\left(2\pi - \sum \theta\right)\right) + k_0 \sum_b \left(V(b)\left(2\pi - \sum \theta\right)\right) - \lambda \sum_c V(c), \tag{1}$$

where the indeces a, b and c run over all space-like edges, time-like edges, and $3-$simplices in the triangulation, respectively. k_0 is related to Newton's constant and λ is related to the cosmological constant. The real functions $V()$ denote the lengths of edges or volumes of tetrahedra, and the sums of θ's tally the dihedral angles subtended by faces meeting at an edge. Inserting explicit expressions for the angles and volumes associated with the simplices of figure 1, the action can be written as

$$\begin{aligned} S &= k_0\left(\frac{2\pi}{i}N_1^S - \frac{2}{i}N_3^{(2,2)}\sin^{-1}\frac{-i2\sqrt{2}\sqrt{2\alpha+1}}{4\alpha+1} - \frac{3}{i}N_3^{(3,1)}\cos^{-1}\frac{-i}{\sqrt{3}\sqrt{4\alpha+1}}\right) \\ &\quad + k_0\sqrt{\alpha}\left(2\pi N_1^T - 4N_3^{(2,2)}\cos^{-1}\frac{-1}{4\alpha+1} - 3N_3^{(3,1)}\cos^{-1}\frac{2\alpha+1}{4\alpha+1}\right) \\ &\quad -\lambda\left(N_3^{(2,2)}\frac{1}{12}\sqrt{4\alpha+2} + N_3^{(3,1)}\frac{1}{12}\sqrt{3\alpha+1}\right). \end{aligned} \tag{2}$$

The variables N_1^S, N_1^T, $N_3^{(3,1)}$ and $N_3^{(2,2)}$ count the number of space-like edges, time-like edges, and $3-$simplicies of the two kinds, respectively. It is possible to manipulate this expression in order to remove the dependence on the number of edges and tetrahedra of each allowed kind from this expression. This leaves S as a function of $N_1 = N_1^T + N_1^S$ and $N_3 = N_3^{(3,1)} + N_3^{(2,2)}$ only.

The model is formulated with Lorentzian signature, but it is possible to perform a Wick rotation into Euclidean space by changing the sign of α. After setting $\alpha = -1$, the action becomes

$$S \to iS_E; \qquad S_E = k_3 N_3 - k_1 N_1 \tag{3}$$

where k_1 and k_3 are

$$k_1 = 2\pi k_0, \qquad k_3 = k_A k_0 + \lambda k_V = \left(6\cos^{-1}\frac{1}{3}\right)k_0 + \lambda\frac{1}{6\sqrt{2}}. \tag{4}$$

The final equation serves as a definition for k_A and k_V. The Euclidean action can also be rewritten as

$$S_E = k_V N_3(\lambda - k_0\zeta) \tag{5}$$

where

$$\zeta = \frac{k_1}{k_0 k_V}\xi - \frac{k_A}{k_V}, \qquad \xi = \frac{N_1}{N_3}. \tag{6}$$

The action is traditionally discussed in terms of the dimensionless parameter ξ. This parameter is known to take a maximal value of 4/3 in purely Euclidean models, but, as is shown below, can only be as high as 5/4 in the Lorentzian version discussed here. Since numerical simulations show that the Euclidean and Lorentzian models have very different properties, one can treat ξ as a suggestive indicator for the large-scale behavior of a model. In a vague sense, ξ can be thought of as measuring the connectivity between simplices. The other parameter ζ, obtained from ξ by a re-scaling and a shift, is introduced for future convenience.

The remaining part of this section is dedicated to estimating the allowed ranges for ζ and ξ and sketching how these ranges affect the partition function of the triangulation model in the large volume limit. To estimate these ranges, one can write down an arbitrary configuration of tetrahedra and define a set of moves, i.e. manipulations of the triangulation, that allow to generate other configurations starting from the initial one. The minimal and maximal values of ξ can be estimated from the way these moves change the configuration of simplices.

It is convenient to keep track of the configuration via a vector f counting the number of simplices of various dimensions,

$$f = f(N_0, N_1^S, N_1^T, N_3). \tag{7}$$

Here N_3 is the sum of $(3,1)$ and $(2,2)$ types of 3−simplices. It is believed that just three types of moves are sufficient to manipulate an initial configuration of simplices into any other one [4]. They are given names like $(a \to b)$, indicating that they change a configuration of a tetrahedra into another configuration with b tetrahedra. For example, one kind of move replaces two $(3,1)$ tetrahedra glued along a common space-like face by a group of six $(3,1)$ simplices (see [4] for details.) The way this and another $(2 \to 3)$ move change the f vector is described by

$$\begin{aligned} \Delta_{(2\to6)} f &= (1,\, 3,\, 2,\, 4), \\ \Delta_{(2\to3)} f &= (0,\, 0,\, 1,\, 1). \end{aligned} \tag{8}$$

Starting with a triangulation consisting of N_1 edges and N_3 tetrahedra, one can construct another configuration by performing x moves of the $(2 \to 6)$ kind and z moves of the $(2 \to 3)$ kind. The total change in the f vector in such a transaction would be

$$\Delta f = (x,\, 3x,\, 2x + z,\, 4x + z), \tag{9}$$

giving a resulting ratio ξ of

$$\xi = \frac{N_1 + 5x + z}{N_3 + 4x + z}. \tag{10}$$

In the limit of large x and z, which corresponds to constructing triangulations of large volume,

$$\lim_{x\to\infty} \xi = \frac{5}{4}, \qquad \lim_{z\to\infty} \xi = 1. \tag{11}$$

These are in fact the upper and lower bounds for the parameter ξ after many substitutions, so the range for ξ is

$$1 \le \xi \le \frac{5}{4}. \tag{12}$$

Using (4) and (6), this can be translated into

$$-9.356 < \zeta < 3.973. \tag{13}$$

The range of ζ plays an important role in determining the partition function of the triangulation model [8]. The partition function is

$$Z(k_0, N_3) = \sum_T W(k_0, N_3, T) e^{-k_V N_3(\lambda - k_0 \zeta)} \tag{14}$$

where $W(k_0, N_3, T)$ is a weighting function that depends on the symmetry of a configuration T and the sum is over all possible configurations having a fixed total number of simplices N_3. In the limit of a large fixed volume $k_V N_3$, the expression for the partition function can be considerably simplified. Up to a factor, Z can be written as

$$Z(k_0, N_3) \sim \sum_T W(k_0, N_3, T) e^{k_V N_3 k_0 \zeta}. \tag{15}$$

The number of triangulations is known to be asymptotically bounded from above by an exponential function [7]. This allows to rewrite the weighting function as

$$W(k_0, N_3, T) = f(k_0, N_3) e^{k_V N_3 s(\zeta)} \tag{16}$$

where $f(k_0, N_3)$ is a function that grows sub-exponentially with N_3, and $s(\zeta)$ is some function that can (in principle) be found using combinatorics. As a result of this substitution, the partition function becomes

$$Z(k_0, N_3) \sim \sum_T f(k_0, N_3) e^{k_V N_3 (s(\zeta) + k_0 \zeta)}. \tag{17}$$

Still working in the large $k_V N_3$ regime, the sum can be replaced by an integral; the integration variable can be taken to be ζ so that

$$Z \sim \int_{\zeta_{min}}^{\zeta_{max}} f(k_0, N_3) e^{k_V N_3 (s(\zeta) + k_0 \zeta)} \, d\zeta. \tag{18}$$

The limits on the integral indicate the minimum and maximum values that the parameter ζ can take. The integral is dominated by the configurations for which the expression $s(\zeta) + k_0 \zeta$ in the exponential is maximized in the allowed range $\zeta_{min} < \zeta < \zeta_{max}$. The position of the maximum depends on the precise form of $s(\zeta)$, but for k_0 large enough, it should occur at ζ_{max}. Contributions to the partition function at other values of ζ are exponentially smaller, so the partition function can be simplified further to

$$Z \sim \int_{\zeta_{min}}^{\zeta_{max}} f(k_0, N_3) e^{k_V N_3 (s(\zeta) + k_0 \zeta)} \delta(\zeta - \zeta_{max}) \, d\zeta = f(k_0, N_3) e^{k_V N_3 (s(\zeta_{max}) + k_0 \zeta_{max})}. \tag{19}$$

The final result is that the macroscopic properties of space-time are determined by the triangulations with $\zeta = 3.973$, or $\xi = 5/4$; these configurations are formed by repetitively applying the $(2 \to 6)$ move in (8).

As mentioned above, the importance of the Lorentzian model is that the upper value $5/4$ for ξ is smaller than the analogous bound of $4/3$ in purely Euclidean triangulation models which are pathological [7]. The correlation between the weaker upper bound for ξ (and ζ) in the Lorentzian model and the observation that the causal dynamical triangulations avoid the non-realistic features of non-causal models provides the hope that the Lorentzian model may describe at least some aspects of quantum gravity.

III. VARIABLE LAPSE

One of the important characteristics of the original Lorentzian model of dynamical triangulations is that all simplices of each type ($(3,1)$ or $(2,2)$) are exactly alike. In particular, all time-like edges have equal length - the model can be said to have a fixed 'lapse.' In this section, the fixed lapse condition is relaxed without compromising the low-energy behavior of the model.

This kind of generalization is known to be possible in $1+1$ dimensions [6]. There, the action is the sum of the areas of the simplices in the triangulation,

$$S^{(2D)} = \lambda \sum_i A \tag{20}$$

and the fixed lapse condition is equivalent to setting all the areas of the triangles to a particular value A_0. The variable lapse generalization consists of considering other triangulations in which the areas of the triangles are not all equal but where the average of these areas is still A_0. These configurations can be viewed as representing the same physical space-time as the fixed-lapse configuration, but implementing a different choice of foliation. Since the actions corresponding to the two types of configurations are equal, the low-energy behavior of the two models is the same.

A similar argument can also be made in higher dimensions. Consider for example working with the action (1) and keeping the assumption that the simplices are either of the type $(3,1)$ or $(2,2)$. Instead of requiring that all the time-like edges have length-squared α as in the original model, suppose that these edge lengths can vary. In other words, tetrahedra may have all time-like edges of equal length or they may have some edges longer than others. The only requirement that has to be imposed is for the faces of neighboring tetrahedra to match so that they can be properly glued together into well-defined triangulation. This requirement always remains implicit and never appears in the equation of the defining action.

In this section, the simplices making up the space-time are labelled by an index v. Each simplex has a volume called V_v. The edges e_j can be either space-like or time-like and each one may have a different length-squared. The types of simplices are distinguished by an index i and the number of simplices of each type are denoted by N_d^i, with d being the dimension. For example, N_1^S is the number of space-like edges and N_3^i is the number of tetrahedra of type i. The dihedral angle at an edge e_j is written as $A(e_j)$ (note, however, that $A(e_j)$ is a function of the length of the edges neighboring e_j as well as to the length of e_j itself.)

The action for the variable-lapse model is of the form given by (1),

$$S_{new} = -k_0\left(\sum_j 2\pi i V(e_j)\right) - \sum_v \left(k_0\left({\sum_j}' V(e_j)A(e_j)\right) + \lambda V_v\right). \tag{21}$$

The sum over j in the first term is over all the edges, both space-like and time-like, in the triangulation. The second (primed) sum over j, however, is restricted to only those edges forming the skeleton of a particular simplex v. Some of the angles $A(e_j)$ are therefore subtended by space-like edges and some are subtended by time-like edges. Also, since space-like and time-like edges are treated together, the volumes $V(e_j)$ corresponding to time-like edges have absorbed factors of i; they can recovered by comparing this action to (1).

There is no constraint in action (21) that forces all the edges to be of equal length, but whenever the simplices can be categorized into particular types, i.e. when several simplices have the same volumes and dihedral angles associated with them, the sums in the above formula can be simplified by grouping repeated terms together. There are at least two suggestive ways to rearrange the terms in the action. One way is to first group the terms arising from each simplex type together and then sum over the types,

$$S_{new} = -k_0\left(\sum_j 2\pi i V(e_j)\right) - \sum_i N_3^i \left(\left(k_0 {\sum_j}' V^i(e_j)A(e_j)\right) + \lambda V^i\right). \tag{22}$$

The second way is to exchange the order of addition. If the cosmological and curvature contributions are collected separately, then

$$S_{new} = -k_0\left(\sum_j 2\pi i V(e_j)\right) - k_0\left(\sum_i N_3^i {\sum_j}' V^i(e_j)A(e_j)\right) - \lambda\left(\sum_i N_3^i V^i\right). \tag{23}$$

Such a model model with variable lapse can be reduced to the original one under some conditions. There are several sets of constraints that are analogous to setting the average area of triangles to A_0 in the $1+1$ dimensional model (20). For example, comparing (23) with (2), it appears natural to set

$$-k_0\left(\sum_j 2\pi i V(e_j)\right) = k_0\left(\frac{2\pi}{i}N_1^S + \sqrt{\alpha_{eff}}\,2\pi N_1^T\right), \tag{24}$$

$$\begin{aligned} -k_0\left(\sum_i N_3^i \sum_{j'} V_i(e_{j'})A(e_{j'})\right) &= -k_0\Bigg(\frac{2}{i}N_3^{(2,2)}\sin^{-1}\frac{-i2\sqrt{2}\sqrt{2\alpha_{eff}+1}}{4\alpha_{eff}+1} + \frac{3}{i}N_3^{(3,1)}\cos^{-1}\frac{-i}{\sqrt{3}\sqrt{4\alpha_{eff}+1}} \\ &\quad +4N_3^{(2,2)}\cos^{-1}\frac{-1}{4\alpha_{eff}+1} + 3N_3^{(3,1)}\cos^{-1}\frac{2\alpha_{eff}+1}{4\alpha_{eff}+1}\Bigg), \end{aligned} \tag{25}$$

$$\lambda\left(\sum_i N_3^i V_i\right) = \lambda\left(N_3^{(2,2)}\frac{1}{12}\sqrt{4\alpha_{eff}+2} + N_3^{(3,1)}\frac{1}{12}\sqrt{3\alpha_{eff}+1}\right). \tag{26}$$

The expressions on the right hand side are terms from the action (2) of the original model with an effective value α_{eff} for the time-like edge-length.

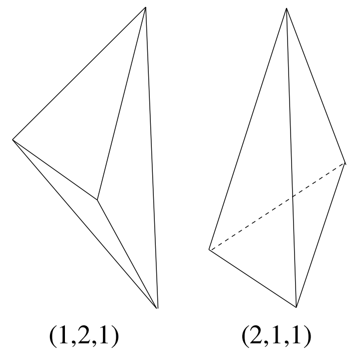

FIG. 2: Simplices in 2+1 dimensions that span two foliation layers.

It is interesting that if the number of simplices in the variable and fixed lapse models are also set to be the same, i.e.

$$\sum_i N_3^i = N_3^{(3,1)} + N_3^{(2,2)}, \qquad \sum_j 1 = N_1^S + N_1^T \tag{27}$$

where the sum over j gives the total number of edges in the triangulation, then (24)-(26) acquire geometrical interpretations. Equation (26), for instance, requires that the average volume of the simplices in the new model be the same as the volume of a simplex in the fixed-lapse model. Equations (24) and (25) require similar averages to hold for edge lengths and dihedral angles. Physically, such conditions would imply that, on the large scale, the variable-lapse and fixed-lapse models would be indistinguishable. One could also impose more stringent conditions such as requiring that the averaging conditions hold true separately on every slice of the triangulation. This would implement a form of time-symmetry that may simplify calculations of, for example, the correlation function between two slices in the space-time.

The form of the action appearing in (22) suggests a different set of averaging conditions: it is possible to have (24) together with the a new condition being the sum of (25) and (26). Adding these last two equation together effectively means that the averaging allows the mixing of the curvature and cosmological contributions to the action; the resulting condition would be weaker than (24)-(26) separately. The effect of this mixing would only have noticeable consequences if an observer had the opportunity to probe the individual simplices of the triangulation.

Do these new models lie in the same equivalence class as the original fixed-lapse model? The answer to this question is in direct analogy with the discussion given in the 1+1 dimensional case [6]. Although the presence of new types of simplices implies that the model has new degrees of freedom, the averaging conditions for the new triangulations freeze some of these new degrees of freedom to the effect that the resulting action can be replaced by an effective fixed-lapse action. Just as in the discussion of the $1+1$ dimensional model, the possible arrangements of lapses in each triangulation or each slice in the triangulation are not summed over. Thus, even if a wider variety of configurations become available under these new conditions, the model stays in the same equivalence class as the fixed-lapse model, in the sense that the partition functions for the two models are equal. It follows that models with varying lapse have the same low energy properties as the original fixed-lapse version.

Similar arguments can be applied in higher dimensions d as well. Conditions analogous to (24)-(27) would then be interpreted as averaging over volumes of d-simplices, or curvatures concentrated on the $d-2$ dimensional faces.

IV. FOLIATIONS WITH PUNCTURES

Another peculiar aspect of the original causal dynamical triangulation model is the arrangement of simplices in a layered structure. In this section, a new model is presented in which the foliation is 'punctured' in the sense that a new type of simplex is allowed to probe multiple layers of tetrahedra. There are several types of tetrahedra that can puncture the foliation in this way: two of them are shown in figure 2. In the $(1,2,1)$ simplex, for example, two vertices are spatially separated from each other but are at the same time in the causal future of the third vertex and in the causal past of the fourth. The foliation, in the sense of the original triangulation model, is punctured by the $(1,2,1)$ tetrahedron because it does not have a space-like face that passes through the spatially separated vertices. Thus, when these simplices are present, it is difficult to construct a space-like hyper-surfaces that span all of the space-time.

The presence of (1,2,1) tetrahedra may affect the large volume behavior of dynamical triangulations. To investigate this possibility, consider a simple model in which there are $(3,1)$, $(2,2)$ and $(1,2,1)$ simplices, all space-like edges have length-squared equal to 1, all time-like edges of the $(3,1)$ and $(2,2)$ tetrahedra and the shorter sides of the $(1,2,1)$ simplex have length-squared $-\alpha$, and the longer time-like edges of the $(1,2,1)$ tetrahedron have length-squared $-\beta$. Expressions for volumes and dihedral angles for the $(1,2,1)$ tetrahedron with this geometry can be computed using the methods of [9]. The volume is

$$V_{(1,2,1)} = \frac{1}{6}\sqrt{\frac{1}{4}\beta^2 - \frac{1}{4}\beta + \alpha\beta} \tag{28}$$

and the dihedral angles θ_S, θ_T and θ_L around the space-like, short time-like and long time-like edges, respectively, are

$$\cos\theta_S = \frac{4\alpha - 2\beta + 1}{4\alpha + 1}, \qquad \cos\theta_T = \frac{\sqrt{\beta}}{\sqrt{4\alpha+1}\sqrt{4\beta+1}}, \qquad \cos\theta_L = \frac{4\alpha + 2 - \beta}{4\alpha - \beta}. \tag{29}$$

The action for this model is an extension of $S_{original}$ in (2) by terms specific to the new simplex type,

$$\begin{aligned} S \;=\; & S_{original} - k_0\left(\frac{1}{i}N_3^{(1,2,1)}\cos^{-1}\frac{(4\alpha-2\beta+1)}{4\alpha+1}\right) - \sqrt{\alpha}k_0\left(4N_3^{(1,2,1)}\cos^{-1}\frac{\sqrt{\beta}}{\sqrt{4\alpha+1}\sqrt{\beta-4\alpha}}\right) \\ & + \sqrt{\beta}k_0\left(2\pi N_1^L - N_3^{(1,2,1)}\cos^{-1}\frac{4\alpha+2-\beta}{4\alpha-\beta}\right) - \lambda\left(N_3^{(1,2,1)}\frac{1}{6}\sqrt{\frac{1}{4}\beta^2 - \frac{1}{4}\beta + \alpha\beta}\right). \end{aligned} \tag{30}$$

The variable N_1^L counts the number of the 'long' time-like edges that have length-squared $-\beta$.

To study the statistical properties of this model, the action should be Wick-rotated into Euclidean space. For the purposes of this paper, this merely means choosing negative values for α and β such that the action becomes purely imaginary. For the part of the action depending on the $(3,1)$ and $(2,2)$ simplices, this can be done by taking $\alpha \to -1$ as before. As for the part of the action specific to the new simplex, a suitable β can be found such that $S \to iS_E$ and S_E is real. After evaluating the angles and volumes, the general form of the action is

$$S_E = \left(\left(k_A k_0 + \lambda k_V\right)N_3^{(3,1)+(2,2)} + \left(k'_A k_0 + \lambda k'_V\right)N_3^{(1,2,1)} - \left(k_1 N_1^{S+T} + k'_1 N_1^L\right)\right), \tag{31}$$

where k_1, k_A, and k_V are the same as in (4). The primed variables can be computed by fixing β. S_E can be written in a form similar to (5) by factorizing the volume terms. The result is

$$S_E = V\left(\lambda - k_0\zeta'\right) \tag{32}$$

where

$$V = k_V N_3^{(3,1)+(2,2)} + k'_V N_3^{(1,2,1)} \tag{33}$$

and

$$\zeta' = \frac{k_1}{k_0}\frac{N_1^{S+T}}{V} - k_A\frac{N_3^{(3,1)+(2,2)}}{V} + \frac{k'_1}{k_0}\frac{N_1^L}{V} - k'_A\frac{N_3^{(1,2,1)}}{V}. \tag{34}$$

Here ζ' is a generalization of ζ in (6).

Since the form of the action S_E is the same as (5), the statistical mechanics of this new model is very similar to that discussed previously. The partition function for the new model,

$$Z'(k_0, N_3) = \sum_T W'(k_0, N_3, T)e^{-k_V N_3(\lambda - k_0\zeta')}, \tag{35}$$

is like (14) but with a larger number of variables summed over. As before, W' should be asymptotically bounded by an exponential in the large-volume limit. The sum can therefore be replaced by an integral over ζ,

$$Z' \sim \int_{\zeta'_{min}}^{\zeta'_{max}} f(k_0, N_3)e^{V(s(\zeta') + k_0\zeta')}\, d\zeta'. \tag{36}$$

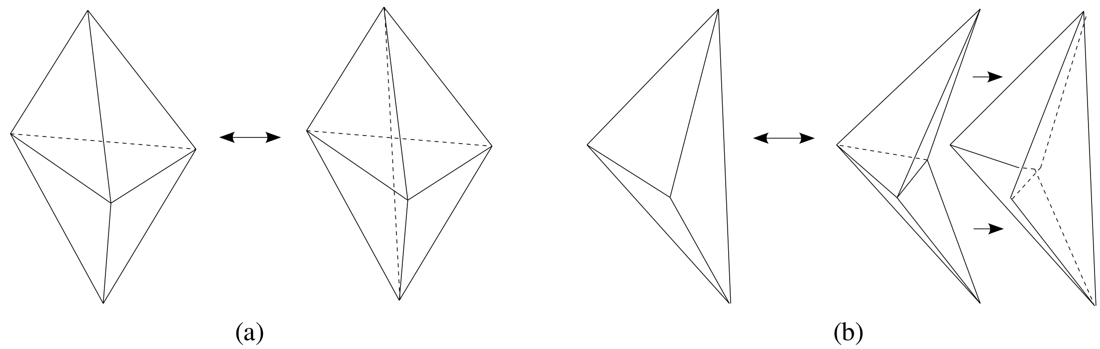

FIG. 3: New moves manipulating $(1,2,1)$ simplices. (a) Two $(3,1)$ change into three $(1,2,1)$ tetrahedra. (b) One $(1,2,1)$ changes into two $(1,2,1)$ and two $(3,1)$ tetrahedra. The $(3,1)$ simplices appear distorted in the diagram.

The dominant contribution to the integral comes from the values of ζ' that maximize the exponent. Under the same assumptions as in section II, one concludes that for k_0 large enough, the partition function should be dominated by maximal values of ζ'. Thus, in that approximation,

$$Z' \sim \int_{\zeta'_{min}}^{\zeta'_{max}} f'(k_0, N_3) e^{V(s(\zeta')+k_0\zeta')} \delta(\zeta' - \zeta'_{max})\, d\zeta' = f'(k_0, N_3) e^{V(s(\zeta'_{max})+k_0\zeta'_{max})}, \tag{37}$$

which is completely analogous to (19).

What is left to explore is how ζ'_{max} differs from ζ_{max}. The governing assumption is that the new parameter ζ'_{max} can serve as an indicator of large-volume behavior just like ξ in the case of the original Euclidean and Lorentzian models. If this assumption is correct, and if the two values ζ_{max} and ζ'_{max} are the same, then the partition functions of the models would differ only by a multiplicative factor and the physics of the two models would be similar. To investigate the range of ζ', one can follow an analogous procedure to that used in section II and explore how various moves affect configurations of tetrahedra.

Because ζ reduces to ξ in the limit $N_3^{(1,2,1)} \to 0$, the moves defined in (8) suggest that values for ζ' within the range (13) should be allowed in the extended model as well. But new moves are now possible as well. In particular, it is important to consider moves that manipulate the number of $(1,2,1)$ tetrahedra. Two such moves are shown in figure 3. The first move is of limited applicability since it cannot be used repeatedly. It does, however, show how to introduce the foliation puncturing simplices into a triangulation that initially contains only $(3,1)$ and $(2,2)$ tetrahedra. The second move can be applied repeatedly and may therefore have an effect on the maximum value of ζ'. It is worth noting that other moves that rearrange simplices while keeping a boundary surface fixed are also possible. However, the two moves shown in the figure are sufficient to discuss the large-scale behavior of the extended model.

The state of the configuration can be described by the vector

$$f' = f'(N_0, N_1^S, N_1^T, N_1^L, N_3^{(3,1)}, N_3^{(2,2)}, N_3^{(1,2,1)}). \tag{38}$$

The two moves shown in the figure, denoted as $(2 \to 3)'$ and $(1 \to 4)'$, change this vector by

$$\begin{aligned} \Delta_{(2\to3)'} f' &= (0,\, 0,\, 0,\, 1,\, -2,\, 0,\, 3), \\ \Delta_{(1\to4)'} f' &= (1,\, 2,\, 2,\, 0,\, 2,\, 0,\, 1). \end{aligned} \tag{39}$$

After a large number of moves of the $(1 \to 4)'$ type, ζ' would be

$$\zeta' = \frac{k_1}{k_0} \frac{4}{2k_V + k'_V} - k_A \frac{2}{2k_V + k'_V} - k'_A \frac{1}{2k_V + k'_V}. \tag{40}$$

The numerical value of expression (40) depends on the value of β, the length-squared of the long edge of the $(1,2,1)$ simplex, which should be between -4 and -1. Such values of β give ζ' in the range $6.42 < \zeta' < 8.4$. and take ζ'_{max} above the Lorentzian level (13), but still keep it below the Euclidean level. Therefore, the new model should be expected to have a different low energy behavior than either the Euclidean model or the original Lorentzian model.

Purely Euclidean models have the undesirable property that the most likely configurations have most of the tetrahedra connected to each other, failing to make up an extended space-time. (The Lorentzian models do not have this

problem.) Since the new move (figure 3 (b)) creates multiple tetrahedra that share the same edge, it is possible that the new Lorentzian model may suffer from a similar effect. It is not immediately clear, however, how strongly the pathological effect would be exhibited in the new model, and one may still hope that some of the desirable properties of the original Lorentzian model are preserved. It is safe to say that the behavior of the extended model should be intermediate between the purely Euclidean and the original Lorentzian cases. To make more precise statements about the low energy behavior, one would need to simulate the extended model on a computer.

As an aside, suppose that for some unknown reason, the frequency of the new simplex type relative to the old types is restricted. This condition would fix the number of moves (39) that would be allowed compared to the number of moves (8). As a consequence, the last two terms in (34) would be fixed, large scale triangulations would be generated by the old moves (8), and ζ' would not violate the Lorentzian bound (13). The triangulations arising from this model would look like configurations in the original model on large scales, but have small bubbles of $(1,2,1)$ tetrahedra on small scales; large clusters of foliation-puncturing tetrahedra would not be present. This kind of restriction on the partition function, however, is admittedly ad-hoc and at the moment does not have a physical or mathematical justification.

V. CONCLUSION

This paper investigated possible ways of relaxing assumptions related to the foliation structure in causal dynamical triangulation models in $2+1$ dimensions. Two results emerged from the discussion.

The first result is that the sizes and shapes of the simplices making up the triangulation can be made variable without compromising the statistical properties of the model. A way to understand this result is to say that when the configurations of simplices are constructed such that the average volumes or angles are the same as in the original fixed-lapse model, the action and therefore the partition function are unchanged. In this view, the requirement for all the simplices to be of the same size and shape is thus traded for a global constraint at the level of the partition function. The argument can be applied in higher dimensions as well.

The second result is concerned with the existence of space-like hyper-surfaces formed from tetrahedral faces. These hyper-surfaces, which appear naturally in the model with only $(3,1)$ and $(2,2)$ tetrahedra, can be punctured by introducing $(1,2,1)$ simplices. The partition function associated with this extended model is found to be different from the partition function of the original model, but it is unclear whether the extended model contains a similar pathology as that exhibited by purely Euclidean models. More precise statements about the large-scale properties of the proposed model (or other significant extensions of the original setup) would require detailed numerical simulations. It may be of some interest, however, that the possibility of a pathology can be removed if the sum over triangulations in the partition function is restricted in certain ways, for example by fixing the ratio of $(1,2,1)$ to the total number of simplices. In other words, allowing a small number of $(1,2,1)$ tetrahedra into a configuration should not spoil the large-scale properties of causal dynamical triangulations.

Acknowledgements. I would like to thank Fotini Markopoulou, Lee Smolin, Mohammad Ansari, and Jan Ambjørn for comments, encouragement and helpful discussions.

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